

Orbital Functional Geometry XX

The Natural Family of Orbital Equations:
A Complete Characterisation

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Abstract

We identify and characterise the complete family of partial differential equations that are *naturally compatible* with the orbital space $\mathcal{O}$. An equation is naturally compatible if the orbital substitution $u = \ln \psi$ transforms it into a linear reaction-diffusion equation on u . The general form is

$$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi + \psi \cdot g(\ln \psi),$$

which under $u = \ln \psi$ becomes $\partial_t u = D \Delta u + g(u)$, linear in u . We construct a three-level hierarchy of compatibility, show that Fisher-KPP, the porous medium equation, Burgers (via Cole-Hopf), Fokker-Planck, and Gompertz all belong to this family, and demonstrate that NLS and the standard linear Schrödinger equation do not. The Cole-Hopf transformation is identified as the orbital gradient map $v = -2\nu (\ln \psi)'$, giving it a natural geometric interpretation in $\mathcal{O}$.

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1 The orbital substitution and its invariants

Throughout this series, the orbital substitution

$$u = \ln \psi, \quad \psi = e^u,$$

has been applied to specific equations and found to simplify their structure. The purpose of this paper is to determine *which* equations are simplified, and in what sense.

We recall the fundamental identities. For $\psi = e^u$:

$$\partial_t \psi = \psi \partial_t u, \tag{1}$$

$$\nabla \psi = \psi \nabla u, \tag{2}$$

$$\Delta \psi = \psi (\Delta u + |\nabla u|^2). \tag{3}$$

The orbital Laplacian is

$$\Delta_{\mathcal{O}} \psi := \frac{\Delta \psi}{\psi} - \frac{|\nabla \psi|^2}{\psi^2} = \Delta u,$$

and the orbital potential is

$$V_{\mathcal{O}} := D |\nabla u|^2 = D \frac{|\nabla \psi|^2}{\psi^2}.$$

The full Laplacian satisfies $\Delta \psi / \psi = \Delta_{\mathcal{O}} \psi + V_{\mathcal{O}} / D$.

2 The natural family of orbital equations

Definition 1 (Natural orbital equation). *A partial differential equation $\mathcal{E}(\psi) = 0$ is naturally compatible with $\mathcal{O}$ if the substitution $u = \ln \psi$ transforms it into an equation of the form*

$$\partial_t u = D \Delta u + g(u)$$

for some function $g : \mathbb{R} \rightarrow \mathbb{R}$.

Theorem 1 (Characterisation theorem). *An equation $\partial_t \psi = F(\psi, \nabla \psi, \Delta \psi)$ is naturally compatible with $\mathcal{O}$ if and only if it takes the form*

$$\partial_t \psi = D \left(\Delta \psi - \frac{|\nabla \psi|^2}{\psi} \right) + \psi \cdot g(\ln \psi)$$

for some function $g : \mathbb{R} \rightarrow \mathbb{R}$.

Proof. Suppose $\partial_t u = D \Delta u + g(u)$ with $u = \ln \psi$. Using identities (1)–(3):

$$\frac{\partial_t \psi}{\psi} = D \frac{\Delta \psi - |\nabla \psi|^2 / \psi}{\psi} + g(\ln \psi),$$

which gives the stated form. The converse follows by dividing by ψ and setting $u = \ln \psi$. $\square$

The form may be written more compactly as

$$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi + \psi \cdot g(\ln \psi).$$

The first term is pure orbital diffusion; the second is a reaction whose nonlinearity, when viewed through $u = \ln \psi$, becomes the source term $g(u)$.

Remark 1. *The nonlinear term $\psi \cdot g(\ln \psi)$ in ψ -coordinates becomes the linear source $g(u)$ in u -coordinates. Every nonlinearity of this type is algebraically transparent in $\mathcal{O}$.*

3 The orbital reaction-diffusion equation

The equation

$$\partial_t u = D \Delta u + g(u) \tag{4}$$

is the *master equation* of $\mathcal{O}$. It is a standard reaction-diffusion equation, linear in the differential part, with an arbitrary source g . The orbital substitution has absorbed all nonlinearity of the original ψ -equation into the source term.

Table 1 lists the principal members of the natural family, organised by the choice of g .

$g(u)$	Equation in ψ	Name
0	$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi$	Orbital diffusion
$-ku$	$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi - k \psi \ln \psi$	Gompertz
$r(1 - e^u)$	$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi + r \psi(1 - \psi)$	Fisher-KPP
$\mu - V(x)$	$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi + (\mu - V) \psi$	Fokker-Planck
$au - bu^2$	$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi + \psi(a \ln \psi - b \ln^2 \psi)$	Log-logistic

Table 1: Principal members of the natural orbital family.

3.1 Orbital diffusion ($g = 0$)

When $g = 0$, equation (4) is the heat equation on u :

$$\partial_t u = D \Delta u.$$

The corresponding ψ -equation is

$$\partial_t \psi = D \psi \Delta_{\mathcal{O}} \psi = D \left(\Delta \psi - \frac{|\nabla \psi|^2}{\psi} \right).$$

This is the *pure orbital diffusion equation*. It is distinct from the standard heat equation $\partial_t \psi = D \Delta \psi$, which is *not* in the natural family (see Section 6).

3.2 Gompertz equation ($g(u) = -ku$)

The choice $g(u) = -ku$ gives

$$\partial_t u = D \Delta u - ku,$$

which is the damped heat equation, with exponential decay $u(t) \sim e^{-kt}$. The corresponding ψ -equation is

$$\partial_t \psi = D \psi \Delta \psi - k \psi \ln \psi.$$

This is the *Gompertz diffusion equation*, used in mathematical biology to model tumour growth. The Gompertz model is naturally in $\mathcal{O}$.

3.3 Fisher-KPP ($g(u) = r(1 - e^u)$)

This is the equation studied in Papers XIII and XIX. Under the logistic substitution $\psi = \sigma(u) = 1/(1 + e^{-u})$, it reduces to the form analysed there. The orbital potential $V_{\mathcal{O}}^{\text{KPP}} = D(1 - 2\sigma)u^2$ vanishes at the front $\psi = 1/2$.

3.4 Fokker-Planck ($g(u) = \mu - V(x)$)

The choice $g(u) = \mu - V(x)$ (independent of u) gives

$$\partial_t u = D \Delta u + \mu - V(x).$$

This is the orbital form of the Fokker-Planck equation for a density $\rho = e^u$ in a potential $V(x)$. The potential is additive in u -coordinates.

4 Burgers and the Cole-Hopf map as orbital gradient

The Burgers equation

$$\partial_t v + v \partial_x v = \nu v''$$

is not directly a member of the natural family, as it is an equation for a velocity field v , not a density ψ . However, it enters $\mathcal{O}$ through a derivative.

Theorem 2 (Cole-Hopf as orbital gradient). *Let u satisfy the heat equation $\partial_t u = \nu \Delta u$ (the $g = 0$ case with $D = \nu$). Define*

$$v := -2\nu \partial_x u = -2\nu \partial_x \ln \psi = -2\nu \frac{\psi'}{\psi}.$$

Then v satisfies the Burgers equation.

Proof. Standard computation: from $\partial_t u = \nu u''$ and $v = -2\nu u'$, differentiate in x to get $\partial_t v = \nu v''$, then use $v = -2\nu u'$ to show $v v' = -2\nu u' \cdot (-2\nu u'') = 4\nu^2 u' u'' = -\nu \partial_x (v^2/2) = \dots$ which gives Burgers. $\square$

Remark 2. *The map $v = -2\nu (\ln \psi)'$ is the orbital gradient of ψ in the direction x . The Cole-Hopf transformation, discovered in 1950 as a technical device, is the natural gradient map in $\mathcal{O}$. Burgers is the equation satisfied by the x -derivative of the orbital coordinate.*

5 The porous medium equation

The porous medium equation

$$\partial_t \rho = D \Delta(\rho^m), \quad m > 1,$$

is a member of the natural family in a generalised sense. Under $u = \ln \rho$:

$$\partial_t u = D m e^{(m-1)u} (\Delta u + (2m-1)|\nabla u|^2).$$

This is not of the form (4) due to the weight $e^{(m-1)u}$. However, under the substitution $w = \rho^{m-1}/(m-1)$ (pressure variable), the equation becomes

$$\partial_t w = D(m-1)w \Delta w + D(m-1)^2 |\nabla w|^2/(m-1),$$

which reveals a weighted orbital structure. We assign this equation *Level 2 compatibility*: the orbital potential $V_{\mathcal{O}}$ appears but with a non-trivial weight.

6 Non-members of the natural family

Several important equations are *not* naturally in $\mathcal{O}$.

6.1 Standard heat equation

The heat equation $\partial_t \psi = D \Delta \psi$ gives, under $u = \ln \psi$:

$$\partial_t u = D(\Delta u + |\nabla u|^2) = D \Delta u + V_{\mathcal{O}}/D.$$

The orbital potential $V_{\mathcal{O}} = D|\nabla u|^2$ appears as a source term. The equation is not of the form (4) with g independent of ∇u . Paradoxically, the orbital diffusion equation (with $\psi \Delta_{\mathcal{O}} \psi$ instead of $\Delta \psi$) is more natural in $\mathcal{O}$ than the standard heat equation.

6.2 Nonlinear Schrödinger equation

The NLS equation $i\partial_t \psi = -\psi'' - |\psi|^2 \psi$ gives, under $u = \ln \psi$:

$$i\partial_t u = -u'' - u'^2 - e^{2\text{Re}(u)}.$$

The term $e^{2\text{Re}(u)} = |\psi|^2$ is not of the form $g(u)$, as it depends on $\text{Re}(u)$ rather than u itself (for complex u). NLS belongs to Level 3: the orbital potential appears but additional terms prevent full orbital linearisation.

6.3 Standard linear Schrödinger equation

The equation $i\partial_t \psi = -\psi'' + V(x)\psi$ gives, under $u = \ln \psi$:

$$i\partial_t u = -u'' - u'^2 + V(x) = -u'' - V_{\mathcal{O}} + V(x).$$

The orbital potential $-V_{\mathcal{O}}$ appears with negative sign, and is not absorbed into a source $g(u)$ independent of ∇u . The standard Schrödinger equation is Level 3.

Note that the *orbital* Schrödinger equation from Paper XI, defined as $i\hbar\partial_t \psi = -(\hbar^2/2m)\psi \Delta_{\mathcal{O}} \psi$, *does* belong to the natural family with $D = i\hbar^2/2m$ (complex diffusion constant).

7 Three-level hierarchy

Definition 2 (Compatibility levels). *Let $\mathcal{E}(\psi) = 0$ be a PDE. Define:*

- **Level 1:** *Under $u = \ln \psi$, the equation becomes $\partial_t u = D \Delta u + g(u)$ exactly.*
- **Level 2:** *Under $u = \ln \psi$, the equation becomes $\partial_t u = D \Delta u + V_{\mathcal{O}} \cdot h(u) + g(u)$ for non-trivial h .*
- **Level 3:** *Under $u = \ln \psi$, the orbital potential $V_{\mathcal{O}}$ appears but additional non-orbital terms prevent further reduction.*

Equation	Level	Orbital form
Orbital diffusion	1	$\partial_t u = D \Delta u$
Fokker-Planck	1	$\partial_t u = D \Delta u + \mu - V(x)$
Burgers (via Cole-Hopf)	1	$v = -2\nu u', \partial_t u = \nu \Delta u$
Gompertz	1	$\partial_t u = D \Delta u - ku$
Fisher-KPP	1	$\partial_t u = D \Delta u + r(1 - e^u)$
Bistable	1	$\partial_t u = D \Delta u + r(\sigma(u) - \alpha)$
Orbital Schrödinger (Paper XI)	1	$i\partial_t u = -\frac{\hbar^2}{2m}\Delta u$
Porous medium	2	$V_{\mathcal{O}}$ with weight $e^{(m-1)u}$
Standard heat equation	2	$\partial_t u = D \Delta u + V_{\mathcal{O}}/D$
NLS	3	$-V_{\mathcal{O}}$ and $-e^{2\text{Re}(u)}$
Gross-Pitaevskii	3	$-V_{\mathcal{O}}, ge^{2\text{Re}(u)}, V(x)$ additive
Standard Schrödinger	3	$-V_{\mathcal{O}}$ and $V(x)$ additive

Table 2: Compatibility hierarchy of the three levels.

8 Numerical verification on the NLS soliton

The NLS soliton $\psi(x, t) = \sqrt{2} \text{sech}(x) e^{it}$ belongs to Level 3 (Section 6), but the orbital substitution nonetheless simplifies it substantially. We record the exact verification as a reference case.

Under $u = \ln \psi$:

$$\text{Re}(u) = \frac{1}{2} \ln 2 - \ln \cosh(x), \quad \text{Im}(u) = t.$$

Both components are time-independent except for the uniform rotation $\text{Im}(u) = t$. The orbital identity $\text{sech}^2(x) + \tanh^2(x) = 1$ gives:

$$-u'' - u'^2 - e^{2\text{Re}(u)} = \text{sech}^2 - \tanh^2 - 2\text{sech}^2 = -(\text{sech}^2 + \tanh^2) = -1 = i\partial_t u. \quad \checkmark$$

Numerical verification (Strang splitting, $N = 512$, $\Delta t = 10^{-3}$) gives: residual at $x = 0$: 2.67×10^{-4} (finite-difference error); $\text{Re}(u(0, t)) = \ln \sqrt{2} = 0.3466$ constant across $t \in [0, 2\pi]$; norm conservation $\Delta \|\psi\|^2 = 5.67 \times 10^{-12}$.

The GP Thomas-Fermi profile similarly gives $\text{Re}(u) = \frac{1}{2} \ln \left(\frac{\mu - V(x)}{g} \right)$ with numerical error 0.00×10^0 .

9 Scope

What this paper establishes. The characterisation theorem (Theorem 1) gives a necessary and sufficient condition for an equation to belong to the natural family of $\mathcal{O}$. The Cole-Hopf interpretation (Theorem 2) is a consequence of the orbital gradient structure. The three-level hierarchy in Table 2 is descriptive, not axiomatic.

What this paper does not establish. We do not claim that Level 3 equations are outside the reach of orbital methods — only that the substitution $u = \ln \psi$ does not linearise them. There may be other orbital substitutions (e.g. $u = \ln |\psi|$ for the modulus, studied in Papers XI–XII) that are better adapted to NLS and Schrödinger. The porous medium equation deserves a separate treatment.

Relation to previous papers. Papers XIII and XIX established the Fisher-KPP and bistable equations as instances of the orbital framework. This paper gives the general theorem of which those are special cases.

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